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AN UNCONDITIONALLY STABLE TIME-STEPPING PROCEDURE WITH ALGORITHMIC DAMPING: A WEIGHTED INTEGRAL APPROACH USING TWO GENERAL WEIGHT FUNCTIONS

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SUMMARY

An accurate algorithm for the integration of the equations of motion arising in structural dynamics is presented. The algorithm is an unconditionally stable single-step implicit algorithm incorporating algorithmic damping. The displacement for a Single-Degree-of-Freedom system is approximated within a time step by a function which is cubic in time. The four coefficients of the cubic are chosen to satisfy the two initial conditions and two weighted integral equations. By considering general weight functions, eight additional coefficients arise. These coefficients are selected to (i) minimize the difference between exact and approximate solutions for small time steps, (ii) incorporate specified algorithmic damping for large time steps, (iii) ensure unconditional stability and (iv) minimize numerical operations in forming the amplification matrix. The accuracy of the procedure is discussed, and the solution time is compared with a widely used algorithm.

KEY WORDS: algorithmic damping; weighted integrals; implicit method; unconditional stability

INTRODUCTION

In structural dynamics, the equations of motion for a spatially discretized system are of the form

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} - \mathbf{F} = 0 \tag{1}$$

where **M**, **C** and **K** are the mass, damping and stiffness matrices, $\ddot{\mathbf{X}}$, $\dot{\mathbf{X}}$ and **X** are acceleration, velocity and displacement vectors, **F** is a prescribed force vector and a dot superscript denotes differentiation with respect to time t. In linear systems, the elements of **M**, **C** and **K** are constant. However, in many important applications **K** is a function of **X** and the problem is nonlinear. In the linear case, if the damping is assumed to be of a restricted type, the equations may be uncoupled into a series of equations, each being of the same form as the equation for a single degree of freedom system, namely

$$m\ddot{x} + c\dot{x} + kx - f = 0 \tag{2}$$

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where m is the mass, c is the damping coefficient, k is the stiffness, x is the displacement and f is the prescribed force.

Many algorithms are available for the numerical integration of equation (1). A number of these are discussed by Hughes and Belytshko[1] and Dokainish and Subbaraj,[2,3] and additional procedures have recently been published.[4–6] A very extensive mathematical treatment is given in the text by Wood.[7] In single-step algorithms, the displacement and velocity are specified at time $t = 0$ and are estimated during the period $0 \leqslant t \leqslant \Delta t$ where Δt is the time step. Values of displacement and velocity are calculated at $t = \Delta t$, and become the initial values for the next time step. In many methods, the acceleration is also considered to be known at the beginning of the time step, and its value at the end of the time step is determined for use as an initial value in the next time step. Many, but by no means all, single-step algorithms approximate displacements within the time step by functions which are cubic in time. These include the general Newmark method,[8] presented in cubic form by Burnett,[9] the Wilson $-\theta$ method[10] and others.[11–13] A number of cubic displacement algorithms use cubic Hermite polynomials as shape functions,[14–16] with the final recurrence relationships being determined using weighted residuals, Hamilton's principle or Hamilton's law of varying action. A detailed description of the weighted residual method using a single weight function is given by Zienkiewicz.[17] In this approach, the weight function is not specified but is defined by what are equivalent to first, second, etc. 'moments of area' of the weight functions in the range $0 \leqslant t \leqslant \Delta t$ about $t = 0$. A number of well-known algorithms are rederived by appropriate selection of these 'moments of area'.

Hilber and Hughes[18] proposed that for a time-stepping algorithm to be competitive, it should possess five attributes, namely it should

(1) be unconditionally stable when applied to linear problems.
(2) involve the solution of no more than one set of implicit equations per time step.
(3) be second-order accurate.
(4) permit controlled algorithmic dissipation of higher modes.
(5) be self-starting.

Free parameters in the Newmark and Wilson methods may be chosen to satisfy the above five attributes. These are probably the most frequently used algorithms in structural engineering applications. In this paper, the authors follow a weighted integral procedure using two general weight functions. Eight parameters which arise during the development are adjusted to satisfy all five attributes, leading to an algorithm which is third or fourth order accurate depending on whether algorithmic damping is included. Comparison with Newmark's method shows the algorithm to be faster for similar accuracy.

WEIGHTED INTEGRAL PROCEDURE

For simplicity, a Single-Degree-Of-Freedom (SDOF) system is considered in detail first, and the extension to Multi-Degree-Of-Freedom (MDOF) systems is then discussed. Lower case letters are used for all matrices in the SDOF case, which are replaced by upper case equivalents when treating MDOF systems. We seek an approximate solution to equation (2) during the time $0 \leqslant t \leqslant \Delta t$ subject to the initial conditions that when $t = 0$, $x = x^0$ and $\dot{x} = \dot{x}^0$. The displacement

within the time step is approximated by the cubic

$$x = [s_{\mathrm{d}}^0(\tau) \quad s_{\mathrm{v}}^0(\tau) \quad s_{\mathrm{d}}^1(\tau) \quad s_{\mathrm{v}}^1(\tau)] \begin{bmatrix} x^0 \\ \dot{x}^0 \Delta t \\ x^1 \\ \dot{x}^1 \Delta t \end{bmatrix} \tag{3}$$

where $\tau = t/\Delta t$ and $s_{\mathrm{d}}^0(\tau)$, etc. are the Hermite polynomials $s_{\mathrm{d}}^0(\tau) = 1 - 3\tau^2 + 2\tau^3$, $s_{\mathrm{v}}^0(\tau) = \tau - 2\tau^2 + \tau^3$, $s_{\mathrm{d}}^1(\tau) = 3\tau^2 - 2\tau^3$ and $s_{\mathrm{v}}^1(\tau) = -\tau^2 + \tau^3$. The superscripts 0 and 1 refer to the non-dimensional times $\tau = 0$ and $\tau = 1$, and the subscripts d and v refer to the influence of displacement and velocity. The following matrices are defined:

$$\mathbf{s}(\tau) = [s_{\mathrm{d}}^0(\tau) \quad s_{\mathrm{v}}^0(\tau) \quad s_{\mathrm{d}}^1(\tau) \quad s_{\mathrm{v}}^1(\tau)] \tag{4}$$

$$\mathbf{d} = \begin{bmatrix} x^0 \\ \dot{x}^0 \Delta t \\ x^1 \\ \dot{x}^1 \Delta t \end{bmatrix} \tag{5}$$

$$\mathbf{t}(\tau) = [1 \quad \tau \quad \tau^2 \quad \tau^3] \tag{6}$$

$$\mathbf{h}_{\mathrm{d}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{h}_{\mathrm{v}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -4 & 6 & -2 \\ 6 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{h}_{\mathrm{a}} = \begin{bmatrix} -6 & -4 & 6 & -2 \\ 12 & 6 & -12 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

The approximate displacement, velocity and acceleration are therefore

$$x = \mathbf{t}(\tau)\mathbf{h}_{\mathrm{d}}\mathbf{d} \quad \dot{x} = \frac{1}{\Delta t}\mathbf{t}(\tau)\mathrm{h}_{\mathrm{v}}\mathbf{d} \quad \ddot{x} = \frac{1}{\Delta t^2}\mathbf{t}(\tau)\mathbf{h}_{\mathrm{a}}\mathbf{d} \tag{8}$$

It is assumed that the force $f = f(\tau)$ within the time step may be approximated by a polynomial

$$f(\tau) \approx \mathbf{t}(\tau)\mathbf{g} \tag{9}$$

where the elements of $\mathbf{g}$ are obtained from the prescribed force. The following approximations for $\mathbf{g}$ may be used.

Linear approximation:

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f^0 \\ f^1 \end{bmatrix} \tag{10}$$

Quadratic approximation:

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f^0 \\ f^{1/2} \\ f^1 \end{bmatrix} \tag{11}$$

Cubic Lagrangian approximation:

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 18 & -45 & 36 & -9 \\ -9 & 27 & -27 & 9 \end{bmatrix} \begin{bmatrix} f^0 \\ f^{1/3} \\ f^{2/3} \\ f^1 \end{bmatrix} \tag{12}$$

Cubic Hermitian approximation:

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} f^0 \\ \dot{f}^0 \Delta t \\ f^1 \\ \dot{f}^1 \Delta t \end{bmatrix} \tag{13}$$

In the above approximations, $f^0, f^{1/3}, f^{1/2}, f^{2/3}$ and f^1 are specified values of the force at $\tau = 0$, $\tau = 1/3$, $\tau = 1/2$, $\tau = 2/3$ and $\tau = 1$, respectively, and $\dot{f}^0$ and $\dot{f}^1$ are derivatives of the force with respect to time at $\tau = 0$, and $\tau = 1$, respectively.

Substituting the values of x, $\dot{x}$ and $\ddot{x}$ from equation (8) and f from equation (9) into equation (2) leads to the residual $r(\tau)$ given by

$$r(\tau) = \left[\frac{\mathrm{m}}{\Delta t^2} \mathbf{t}(\tau) \mathbf{h}_\mathrm{a} + \frac{c}{\Delta t} \mathbf{t}(\tau) \mathbf{h}_\mathrm{v} + k\, \mathbf{t}(\tau) \mathbf{h}_\mathrm{d} \right] \mathbf{d} - \mathbf{t}(\tau)\, \mathbf{g} \tag{14}$$

Carrying out the multiplication gives

$$r(\tau) = \frac{1}{\Delta t^2} \mathbf{t}(\tau) [\overline{\mathbf{m}} \mathbf{d} - \Delta t^2 \mathbf{g}] \tag{15}$$

where

$$\overline{\mathbf{m}} = \begin{bmatrix} -6m + \Delta t^2 k & -4m + \Delta tc & 6m & -2m \\ 12m - 6\Delta tc & 6m - 4\Delta tc + \Delta t^2 k & -12m + 6\Delta tc & 6m - 2\Delta tc \\ 6\Delta tc - 3\Delta t^2 k & 3\Delta tc - 2\Delta t^2 k & -6\Delta tc + 3\Delta t^2 k & 3\Delta tc - \Delta t^2 k \\ 2\Delta t^2 k & \Delta t^2 k & -2\Delta t^2 k & \Delta t^2 k \end{bmatrix} \tag{16}$$

We now consider the two weighted integrals

$$r_i^{\mathrm{w}} = \int_0^1 r(\tau) W_i(\tau)\, \mathrm{d}\tau,\ i = 1, 2 \tag{17}$$

where $W_i(\tau)$, $i = 1, 2$ are unspecified weight functions. Define

$$\mathbf{u}_i = \int_0^1 \mathbf{t}(\tau) W_i(\tau)\, \mathrm{d}\tau = [u_{i0} \quad u_{i1} \quad u_{i2} \quad u_{i3}],\ i = 1, 2 \tag{18}$$

where

$$u_{ij} = \int_0^1 \tau^j W_i(\tau)\, \mathrm{d}\tau, \quad i = 1, 2, \quad j = 0, 1\ , 2, 3 \tag{19}$$

The weighted integrals then become

$$\Delta t^2\, \mathbf{r}^{\mathrm{w}} = \mathbf{u}(\overline{\mathbf{m}}\, \mathbf{d} - \Delta t^2 \mathbf{g}) = \mathbf{p}\mathbf{d} - \Delta t^2 \mathbf{u}\mathbf{g} \tag{20}$$

where

$$\mathbf{r}^{\mathrm{w}} = [r_1^{\mathrm{w}} \quad r_2^{\mathrm{w}}]^{\mathrm{T}} \tag{21}$$

and

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \end{bmatrix} \tag{22}$$

Partitioning $\mathbf{p}$ and $\mathbf{d}$ as

$$\mathbf{p} = [\mathbf{p}^0 \quad \mathbf{p}^1] \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}^0 \\ \mathbf{d}^1 \end{bmatrix} \tag{23}$$

equation (20) then becomes

$$\Delta t^2\, r^{\mathrm{w}} = [\mathbf{p}^0 \quad \mathbf{p}^1] \quad \begin{bmatrix} \mathbf{d}^0 \\ \mathbf{d}^1 \end{bmatrix} - \Delta t^2\, \mathbf{u}\mathbf{g} \tag{24}$$

We now set

$$\mathbf{r}^{\mathrm{w}} = \mathbf{0} \tag{25}$$

and solve equation (24) for $\mathbf{d}^1$ giving

$$\mathbf{d}^1 = -(\mathbf{p}^1)^{-1}\mathbf{p}^0\mathbf{d}^0 + \Delta t^2(\mathbf{p}^1)^{-1}\mathbf{u}\mathbf{g} = \mathbf{a}^{\mathrm{a}}\mathbf{d}^0 + \mathbf{b}\mathbf{g} \tag{26}$$

where $\mathbf{a}^{\mathrm{a}} = -(\mathbf{p}^1)^{-1}\,\mathbf{p}^0$ is the approximate amplification matrix and $\mathbf{b} = \Delta t^2\,(\mathbf{p}^1)^{-1}\,\mathbf{u}$.

EXACT SOLUTION FOR THE UNFORCED CASE

In the following sections, underdamped or critically damped free vibration is assumed, i.e. $f = 0$ and $\zeta \leqslant 1$ where $\zeta = c/\sqrt{4km}$. The results are however valid for the overdamped case. The exact solution of equation (2) is readily determined as

$$x = \frac{\mathrm{e}^{-\zeta\omega t}}{\Omega_{\mathrm{d}}}[\Omega_{\mathrm{d}}\cos\omega_{\mathrm{d}}t + \zeta\Omega\sin\omega_{\mathrm{d}}t \quad \sin\omega_{\mathrm{d}}t]\begin{bmatrix} x_0 \\ \dot{x}_0\,\Delta t \end{bmatrix} \tag{27}$$

where $\omega = \sqrt{k/m}$, $\omega_{\mathrm{d}} = \omega\sqrt{1-\zeta^2}$, $\Omega = \omega\,\Delta t$, $\Omega_{\mathrm{d}} = \omega_{\mathrm{d}}\,\Delta t$ and Δt is the time step used in the approximate procedure. ζ is the damping ratio. The exact amplification matrix $\mathbf{a}^{\mathrm{e}}$ is then

$$\mathbf{a}^{\mathrm{e}} = \begin{bmatrix} a^{\mathrm{e}}_{11} & a^{\mathrm{e}}_{12} \\ a^{\mathrm{e}}_{21} & a^{\mathrm{e}}_{22} \end{bmatrix} = \frac{\mathrm{e}^{-\zeta\Omega}}{\Omega_{\mathrm{d}}}\begin{bmatrix} \Omega_{\mathrm{d}}\cos\Omega_{\mathrm{d}} + \zeta\Omega\sin\Omega_{\mathrm{d}} & \sin\Omega_{\mathrm{d}} \\ -\Omega^2\sin\Omega_{\mathrm{d}} & \Omega_{\mathrm{d}}\cos\Omega_{\mathrm{d}} - \zeta\Omega\sin\Omega_{\mathrm{d}} \end{bmatrix} \tag{28}$$

with $\mathbf{a}^{\mathrm{e}}$ having the Jordan form

$$\mathbf{j}^{\mathrm{e}} = \mathrm{e}^{-\zeta\Omega}\begin{bmatrix} \cos\Omega_{\mathrm{d}} + \mathrm{i}\sin\Omega_{\mathrm{d}} & 0 \\ 0 & \cos\Omega_{\mathrm{d}} - \mathrm{i}\sin\Omega_{\mathrm{d}} \end{bmatrix} \tag{29}$$

APPROXIMATE SOLUTION FOR THE UNFORCED CASE

The approximate amplification matrix for the unforced case, determined using equation (26), may be expressed as

$$\mathbf{a}^{\mathrm{a}} = \begin{bmatrix} a^{\mathrm{a}}_{11} & a^{\mathrm{a}}_{12} \\ a^{\mathrm{a}}_{21} & a^{\mathrm{a}}_{22} \end{bmatrix} \tag{30}$$

which has the Jordan form

$$\mathbf{j}^{\mathrm{a}} = \begin{bmatrix} \alpha + \mathrm{i}\beta & 0 \\ 0 & \alpha - \mathrm{i}\beta \end{bmatrix} \tag{31}$$

where

$$\alpha = (a^{a}_{11} + a^{a}_{22})/2 \tag{32}$$

and

$$\beta^2 = -((a^{a}_{11} - a^{a}_{22})^2 + 4a^{a}_{12}a^{a}_{21})/4 \tag{33}$$

In the SDOF case, it is convenient to use the notation used in the exact solution. Thus the definitions following equation (27) lead to alternative expressions for c and k, namely $c = 2m\zeta\Omega/\Delta t$ and $k = m\Omega^2/\Delta t^2$. The elements a^{a}_{ij}, $i = 1, 2$ $j = 1, 2$ of $\mathbf{a}^{a}$ are then of the form

$$a^{a}_{ij} = a^{a}_{ij}(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23}, \zeta, \Omega) \tag{34}$$

It should also be noted that for time steps which are large compared with the period of the system, i.e. as $\Omega \to \infty$,

$$a^{a}_{ij} = a^{a}_{ij}(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23}, \zeta, \infty) = a^{a}_{ij}(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23}) \tag{35}$$

which are independent of the damping ratio ζ.

SPECTRAL RADIUS

The spectral radius, ρ, is the larger absolute value of the eigenvalues of the amplification matrix, which are the diagonal terms of the Jordan form. With the exact solution the spectral radius $\rho^{e} = e^{-\zeta\Omega}$, which is $\leqslant 1$ for all ζ and Ω.

Three cases arise when considering the approximate Jordan form of $\mathbf{a}^{a}$.

(1) If $\beta^2 \geqslant 0$, $\rho^{a} = \sqrt{\alpha^2 + \beta^2}$.
(2) If $\beta^2 < 0$ and $\alpha \leqslant 0$, $\rho^{a} = -\alpha + \sqrt{-\beta^2}$.
(3) If $\beta^2 < 0$ and $\alpha > 0$, $\rho^{a} = \alpha + \sqrt{-\beta^2}$.

From equations (29) and (31), it is clear that for $\mathbf{j}^{a}$ to be a reasonable approximation of $\mathbf{j}^{e}$ in the underdamped case, β must be real, and hence we require $\beta^2 \geqslant 0$. In all cases,

$$\rho^{a} = \rho^{a}(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23}, \zeta, \Omega) \tag{36}$$

For unconditional stability, $\rho^{a} \leqslant 1$ for all Ω.

SELECTION OF THE COEFFICIENTS u_{ij}

The eight coefficients u_{ij}, $i = 1, 2$ $j = 0, 1, 2, 3$ are chosen in a series of steps, which involve minimizing errors, ensuring unconditional stability and optimizing numerical efficiency.

Step 1. *Eliminate* u_{22} *using error minimization*

The error $|\varepsilon|$ is defined using the exact and approximate Jordan forms of the amplification matrix as

$$\varepsilon^2 = (\mathrm{e}^{-\zeta\Omega}\cos\Omega_\mathrm{d} - \alpha)^2 + (\mathrm{e}^{-\zeta\Omega}\sin\Omega_\mathrm{d} - \beta)^2 \tag{37}$$

subject to the constraint $\beta^2 \geqslant 0$. The error $|\varepsilon|$ is seen on an argand diagram as the distance between the approximate and exact coordinates representing the eigenvalues of the amplification matrices.

Expanding ε^2 in equation (37) as a Taylor's series in Ω we obtain

$$\varepsilon^2 = f_1(u_{10}, u_{11}, u_{12}, u_{20}, u_{21}, u_{22})(\Omega^6 + \zeta f_2(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23})\Omega^7) + O(\Omega^8) \tag{38}$$

where

$$f_1 = \frac{1}{1-\zeta^2}\left(\frac{6(u_{12}u_{21} - u_{22}u_{11}) + 3(u_{10}u_{22} - u_{20}u_{12}) + 2(u_{11}u_{20} - u_{21}u_{10})}{24(u_{11}u_{20} - u_{21}u_{10})}\right)^2.$$

Setting the coefficient of Ω^6 to zero and solving for u_{22} we obtain

$$u_{22} = \frac{6u_{12}u_{21} - 3u_{12}u_{20} + 2u_{11}u_{20} - 2u_{10}u_{21}}{3(2u_{11} - u_{10})} \tag{39}$$

As f_1 is also a factor of the coefficient of Ω^7, after this substitution of u_{22}, $\varepsilon^2 = O(\Omega^8)$.

Step 2. *Eliminate* u_{23} *by introducing an algorithmic damping factor*

Substituting u_{22} in equation (39) into the equation

$$\rho^\mathrm{a} = \sqrt{\alpha^2 + \beta^2} \tag{40}$$

and taking the limit as $\Omega \to \infty$ we obtain

$$\lim_{\Omega\to\infty} \rho^\mathrm{a} = \rho^\infty = \rho^\mathrm{a}(u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{23}) \tag{41}$$

which is independent of ζ. To provide algorithmic dissipation of higher frequencies, a value of ρ^∞ of between 0·5 and 0·8 has been suggested.[7] Taking $\bar{\rho}$ as the specified value of ρ^∞ and solving equation (41) for u_{23} we obtain

$$\begin{aligned} u_{23} = {} & (3u_{13}(2u_{11} - u_{10})(2u_{21} - u_{20}) - (3u_{12} - u_{10})(u_{10}u_{21} - u_{11}u_{20}) \\ & - u_{13}(6u_{12}u_{21} - 3u_{12}u_{20} + 2u_{11}u_{20} - 2u_{10}u_{21})(1 - \bar{\rho}^2))/ \\ & (3(2u_{11} - u_{10})(2u_{11} - u_{10} - u_{12}(1 - \bar{\rho}^2))) \end{aligned} \tag{42}$$

Substitution of u_{23} from equation (42) into the expressions for α and β^2 we find these are now independent of u_{20} and u_{21} in addition to u_{22} and u_{23}. Thus with the substitutions for u_{22} and

u_{23} we now find

$$\alpha = \alpha(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho}, \Omega) \tag{43}$$

$$\beta^2 = \beta^2(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho}, \Omega) \tag{44}$$

$$\varepsilon^2 = \varepsilon^2(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho}, \Omega) \tag{45}$$

Step 3. *Select* u_{12} *and* u_{13} *to ensure* $\beta^2 \geqslant 0$

Although there may be other alternatives, at this stage we seek combinations of u_{10}, u_{11}, u_{12} and u_{13} for which β^2 is a perfect square, which ensures $\beta^2 \geqslant 0$ for all Ω. β^2 in equation (44) may be written in the form

$$\begin{aligned}\beta^2 = \Omega_{\mathrm{d}}^2 (&f_6(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho})\Omega^6 + \zeta f_5(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho})\Omega^5 \\ &+ f_4(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho})\Omega^4 + \zeta f_3(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho})\Omega^3 \\ &+ f_2(u_{10}, u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho})\Omega^2 + \zeta f_1(u_{10}, u_{11}, u_{12}, \bar{\rho})\Omega \\ &+ f_0(u_{10}, u_{11}, u_{12}, \bar{\rho}))/(f(u_{11}, u_{12}, u_{13}, \zeta, \bar{\rho}, \Omega))^2\end{aligned} \tag{46}$$

We now express the sixth-order polynomial in Ω in brackets in equation (46) as the perfect square $(c_3\Omega^3 + c_2\Omega^2 + c_1\Omega + c_0)^2$. Equating coefficients in powers of Ω leads to seven non-linear equations in the variables $u_{10}, u_{11}, u_{12}, u_{13}, c_0, c_1, c_2$ and c_3. Seven solution sets arise, with only one valid set relating u_{10}, u_{11}, u_{12} and u_{13}, namely

$$u_{12} = q_1 u_{10} + q_2 u_{11} \quad \text{and} \quad u_{13} = q_3 u_{10} + q_4 u_{11} \tag{47}$$

where u_{10} and u_{11} are arbitrary and

$$q_1 = \frac{1}{3(1+\bar{\rho})}, \quad q_2 = \frac{2(2+\bar{\rho})}{3(1+\bar{\rho})}, \quad q_3 = -\frac{(2+\bar{\rho})}{3(1+\bar{\rho})^2}, \quad q_4 = \frac{5+5\bar{\rho}+2\bar{\rho}^2}{3(1+\bar{\rho})^2} \tag{48}$$

Non-valid solutions are either trivial, or are infinite if $\bar{\rho} = 1$. Backsubstituting u_{12} and u_{13} from equation (47) into equations (39) and (42) gives the solution set

$$u_{22} = q_1 u_{20} + q_2 u_{21} \quad \text{and} \quad u_{23} = q_3 u_{20} + q_4 u_{21} \tag{49}$$

with u_{20} and u_{21} arbitrary. The form of these coefficients is similar to the form of u_{12} and u_{13} given in equation (47), which is clearly a requirement, since the same results should be obtained by reversing the weight functions.

The final forms for α, β and ε^2 are obtained by back substituting u_{12} and u_{13} from equation (47) into equations (43)–(45) giving

$$\begin{aligned}&\alpha(\zeta, \bar{\rho}, \Omega) \\ &= \frac{\bar{\rho}\Omega^4 - 2\zeta(1-\bar{\rho}^2)\Omega^3 - 2((7-6\zeta^2)(1+\bar{\rho})^2 + 2\bar{\rho})\Omega^2 + 12\zeta(1-\bar{\rho}^2)\Omega + 36(1+\bar{\rho})^2)}{\Omega^4 + 4\zeta(2+\bar{\rho})\Omega^3 + 4(1+\bar{\rho}+\bar{\rho}^2+6\zeta^2(1+\bar{\rho}))\Omega^2 + 24\zeta(1+\bar{\rho})(2+\bar{\rho})\Omega + 36(1+\bar{\rho})^2}\end{aligned} \tag{50}$$

$$\beta(\zeta, \bar{\rho}, \Omega) = \frac{-2\Omega_d((1+4\bar{\rho}+\bar{\rho}^2)\Omega^2 - 6\zeta(1-\bar{\rho}^2)\Omega - 18(1+\bar{\rho}))^2}{\Omega^4 + 4\zeta(2+\bar{\rho})\Omega^3 + 4(1+\bar{\rho}+\bar{\rho}^2+6\zeta^2(1+\bar{\rho}))\Omega^2 + 24\zeta(1+\bar{\rho})(2+\bar{\rho})\Omega + 36(1+\bar{\rho})^2} \tag{51}$$

$$\varepsilon^2(\zeta, \bar{\rho}, \Omega) = \left(\frac{1-\bar{\rho}}{72(1+\bar{\rho})}\right)^2 \Omega^8 - \frac{\zeta(1-\bar{\rho})(19-2\bar{\rho}-11\bar{\rho}^2)}{38880(1+\bar{\rho})^3}\Omega^9 - \frac{(17\bar{\rho}^4 - 17\bar{\rho}^3 - 18\bar{\rho}^2 - 17\bar{\rho} + 17) - 5\zeta^2(1-\bar{\rho})(31\bar{\rho}^3 + 33\bar{\rho}^2 - 57\bar{\rho} - 79)}{583200(1+\bar{\rho})^4}\Omega^{10} + O(\Omega^{11}) \tag{52}$$

From the Jordan forms of the exact and approximate amplification matrices, $\beta(\zeta, \bar{\rho}, \Omega)$ should approximate $e^{-\zeta\Omega}\sin\Omega_d$ for small Ω. Comparison of these expressions indicates that the negative root of β^2 is the correct root to adopt. The error $|\varepsilon|$ is now independent of the remaining parameters u_{10}, u_{11}, u_{20} and u_{21}, and no further analytical reduction of the error is possible. Their selection based on numerical aspects is considered later.

The approximate amplification matrix now reduces to the simple and elegant form

$$\mathbf{a}^a = \frac{1}{\Omega_d}\begin{bmatrix} \Omega_d\,\alpha(\zeta, \bar{\rho}, \Omega) + \zeta\,\Omega\,\beta(\zeta, \bar{\rho}, \Omega) & \beta(\zeta, \bar{\rho}, \Omega) \\ -\Omega^2\ \beta(\zeta, \bar{\rho}, \Omega) & \Omega_d\,\alpha(\zeta, \bar{\rho}, \Omega) - \zeta\,\Omega\,\beta(\zeta, \bar{\rho}, \Omega) \end{bmatrix} \tag{53}$$

where $\alpha(\zeta, \bar{\rho}, \Omega)$ and $\beta(\zeta, \rho, \Omega)$ are given by equations (50) and (51). The terms of the approximate amplification matrix are of a similar form to those of the exact amplification matrix given in equation (28), with $\alpha(\zeta, \bar{\rho}, \Omega)$ corresponding to $e^{-\zeta\Omega}\cos\Omega d$ and $\beta(\zeta, \bar{\rho}, \Omega)$ corresponding to $e^{-\zeta\Omega}\sin\Omega_d$.

If $\bar{\rho} = 1$, α, β and $|\varepsilon|$ become

$$\alpha(\zeta, 1, \Omega) = \frac{\Omega^4 - 12(5-4\zeta^2)\Omega^2 + 144}{\Omega^4 + 2\zeta\Omega^3 + 12(1+4\zeta^2)\Omega^2 + 144\zeta\Omega + 144} \tag{54}$$

$$\beta(\zeta, 1, \Omega) = \frac{12(12-\Omega^2)\Omega\sqrt{(1-\zeta^2)}}{\Omega^4 + 2\zeta\Omega^3 + 12(1+4\zeta^2)\Omega^2 + 144\zeta\Omega + 144} \tag{55}$$

$$|\varepsilon(\zeta, 1, \Omega)| = \frac{1}{720}\Omega^5 + O(\Omega^6) \tag{56}$$

Thus if $\bar{\rho} = 1$ the algorithm is fourth-order accurate. If algorithmic damping is considered, i.e. $\bar{\rho} \neq 1$, the error in $|\varepsilon|$ is $O(\Omega^4)$, and the algorithm is third-order accurate. Hence the introduction of algorithmic damping inevitably reduces the accuracy.

It should also be noted that if $\zeta > 1$, i.e. the system is overdamped, β becomes complex, which is consistent with the exact solution. Repeating the above procedure for the overdamped case, with the trigonometric terms in equation (37) being replaced by hyperbolic equivalents, leads to the same expressions for α and $|\varepsilon|$, and in the expression for β, $\sqrt{(1-\zeta^2)}$ is replaced by $i\sqrt{(\zeta^2-1)}$.

Step 4. *Select* u_{11} *and* u_{21} *for numerical efficiency*

When determined analytically, the amplification matrix (equation (53)) is independent of the remaining coefficients u_{10}, u_{11}, u_{20} and u_{21}. In practice, the amplification matrix is determined numerically by solving the MDOF equivalent of

$$\mathbf{p}^1 \mathbf{a}^a = -\mathbf{p}^0 \tag{57}$$

which requires the decomposition of $\mathbf{p}^1$. There are significant computational savings if $\mathbf{p}^1$ is symmetrical. Equating coefficients of powers of Ω in p^1_{12} and p^1_{21} it is found that $\mathbf{p}^1$ is symmetrical for all ζ and Ω if

$$u_{11} = \frac{2+\bar{\rho}}{3(1+\bar{\rho})} u_{10} + u_{20} \quad \text{and} \quad u_{21} = -\frac{1}{6(1+\bar{\rho})} u_{10} \tag{58}$$

with u_{10} and u_{20} arbitrary.

Step 5. *Select* u_{20} *for numerical conditioning*

The coefficient u_{20} may be selected so that in the case of very small time steps, that is as $\Omega \to 0$, the matrix $\mathbf{p}^1$ becomes diagonal. Setting $p^1_{12} = p^1_{21} = 0$ and solving for u_{20} we obtain

$$u_{20} = -\frac{u_{10}}{3(1+\bar{\rho})} \tag{59}$$

Step 6. *Select* u_{10} *for simplicity*

The only remaining coefficient is u_{10}, which scales the weight functions, and hence any convenient value may be selected. Terms of $\mathbf{u}$ without denominators arise if we take

$$u_{10} = 18(1+\bar{\rho})^2 \tag{60}$$

leading to the final recommended matrix of coefficients

$$\mathbf{u} = \begin{bmatrix} 18(1+\bar{\rho})^2 & 6(1+\bar{\rho})^2 & 2(1+\bar{\rho})(1+2\bar{\rho}) & -2(1-2\bar{\rho}-2\bar{\rho}^2) \\ -6(1+\bar{\rho}) & -3(1+\bar{\rho}) & -2(1+\bar{\rho}) & -(1+2\bar{\rho}) \end{bmatrix} \tag{61}$$

With no algorithmic damping, i.e. if $\bar{\rho} = 1$, the values are

$$\mathbf{u} = \begin{bmatrix} 72 & 24 & 12 & 6 \\ -12 & -6 & -4 & -3 \end{bmatrix} \tag{62}$$

Referring to equations (22) and (23), the elements of matrices $\mathbf{p}^0$ and $\mathbf{p}^1$ are finally obtained as

$$p^0_{11} = -36(1+\bar{\rho})^2 m - 12(1+\bar{\rho})(2+\bar{\rho})\Delta t c + 2(4+13\bar{\rho}+7\bar{\rho}^2)\Delta t^2 k \tag{63}$$

$$p^0_{12} = -36(1+\bar{\rho})^2 m + 6\bar{\rho}(1+\bar{\rho})\Delta t c + 2\bar{\rho}(2+\bar{\rho})\Delta t^2 k \tag{64}$$

$$p^0_{21} = 6(1+\bar{\rho})\Delta t c - 2(1+2\bar{\rho})\Delta t^2 k \tag{65}$$

$$p^0_{22} = 6(1+\bar{\rho}) m - \bar{\rho}\Delta t^2 k \tag{66}$$

$$p^1_{11} = 36(1+\bar{\rho})^2 m + 12(1+\bar{\rho})(2+\bar{\rho})\Delta t c + 2(5+5\bar{\rho}+2\bar{\rho}^2)\Delta t^2 k \tag{67}$$

$$p^1_{12} = p^1_{21} = -6(1+\bar{\rho})\Delta t c - 2(2+\bar{\rho})\Delta t^2 k \tag{68}$$

$$p^1_{22} = -6(1+\bar{\rho}) m + \Delta t^2 k \tag{69}$$

Table I. Relative period elongation

$\frac{\Delta t}{T^e}$	Proposed method			Trapezoidal rule
	$\bar{\rho} = 1$	$\bar{\rho} = 0{\cdot}9$	$\bar{\rho} = 0{\cdot}8$	
0.00	0·000000	0·000000	0·000000	0·000000
0·05	0·000013	0·000014	0·000014	0·008171
0·10	0·000211	0·000212	0·000216	0·032075
0·15	0·001039	0·001044	0·001061	0·070085
0·20	0·003151	0·003166	0·003220	0·120033
0·25	0·007294	0·007330	0·007454	0·179677
0·30	0·014181	0·014251	0·014490	0·247004
0·35	0·024377	0·024493	0·024893	0·320344
0·40	0·038231	0·038404	0·039004	0·398381

The elements of **b** are not presented in algebraic form, as they are obtained numerically in practice using the decomposition of $\mathbf{p}^1$ when the amplification matrix is determined.

MDOF SYSTEMS

Extension of the procedure described for SDOF cases in the previous sections to allow MDOF cases to be analysed is relatively simple. The scalar quantities m, c and k in equations (63)–(69) are replaced by structural mass, damping and stiffness matrices **M**, **C** and **K**, leading to matrices $\mathbf{P}^0_{11} - \mathbf{P}^1_{22}$ which are the MDOF equivalents of $p^0_{11} - p^1_{22}$. The matrices $\mathbf{P}^0_{11} - \mathbf{P}^1_{22}$ are symmetrical with the same banding as **M**, **C** and **K**.

COMPARISON WITH OTHER ALGORITHMS

Period elongation and amplitude decay

Two features are frequently used to compare algorithms, namely period elongation and amplitude decay. Hilber and Hughes[18] conducted an extensive study of time-stepping algorithms, and compared relative period errors associated with the Newmark,[8] Wilson,[10] Houbolt,[19] Park[20] and α[21] methods. They showed that of these algorithms, the trapezoidal rule, which is identical to the Newmark method with $\beta = 1/4$ and $\gamma = 1/2$, produced the lowest period elongation. The trapezoidal rule, which is unconditionally stable, does not produce algorithmic damping, and has $\rho = 1$ for all Ω.

Results obtained using the proposed method were compared with those obtained using the trapezoidal rule. In Table I, relative period elongation $(T^a - T^e)/T^e$, where T^a and T^e are approximate and exact periods obtained using the proposed algorithm for three values of $\bar{\rho}$ are shown and compared with the trapezoidal rule for various time steps. Recall $\Omega = \omega\Delta t$, or in terms

Table II. Spectral radius ρ

$\frac{\Delta t}{T^e}$	Proposed method			Trapezoidal rule
	$\bar{\rho} = 1$	$\bar{\rho} = 0{\cdot}9$	$\bar{\rho} = 0{\cdot}8$	
0·05	1·000000	0·999993	0·999985	1·000000
0·10	1·000000	0·999890	0·999767	1·000000
0·20	1·000000	0·998449	0·996658	1·000000
0·30	1·000000	0·993356	0·986042	1·000000
0·40	1·000000	0·983968	0·966524	1·000000
0·50	1·000000	0·971929	0·941816	1·000000
1·00	1·000000	0·927407	0·853052	1·000000
2·00	1·000000	0·907231	0·813905	1·000000
4·00	1·000000	0·901812	0·803480	1·000000
8·00	1·000000	0·900453	0·800869	1·000000

of the period, T, $\Omega = 2\pi\Delta t/T$. The relative period elongation is $O(\Delta t/T)^4$ using the proposed method, and there is little decrease in accuracy due to the inclusion of algorithmic damping. The proposed method leads to much greater accuracy than the trapezoidal rule. In Table II, the spectral radius is given for three values of $\bar{\rho}$, and for interest the unit value obtained using the trapezoidal rule is also shown. The spectral radius remains approximately unity for time steps up to $\Delta t/T^e = 0.2$, and of course approaches the specified value, $\bar{\rho}$, for large time steps.

Solution time comparisons

A range of parameters arise in comparing the numerical efficiency of various techniques, including number of degrees of freedom, band width of matrices, number of time steps, required accuracy and whether algorithmic damping is present. The proposed method was compared with the Newmark's method, using the single-step two-stage version of the algorithm.[7] The general forms of the two methods were programmed, that is versions in which the Newmark parameters β and γ and the weighted integral parameter $\bar{\rho}$ could be varied. Neglecting the time taken to generate mass, damping and stiffness matrices, the solution times for Newmark's method and the proposed method may be expressed as

$$t_N = t_N^A + t^T N_N \quad \text{and} \quad t_W = t_W^A + t^T N_W \tag{70}$$

respectively, where t_N^A and t_W^A are the times to generate the amplification matrix, N_N and N_W are the number of time steps, and t^T is the time per time step, which is the same with both methods. For similar accuracy, $N_W < N_N$ as discussed below.

Applying the same definition of the error $|\varepsilon|$ used with the proposed method to Newmark's method with $\beta = 1/4$ and $\gamma = 1/2$, we obtain

$$|\varepsilon(\zeta, 1, \Omega)| = \frac{1}{12}\Omega^3 + O(\Omega^4) \tag{71}$$

Table III. Time-step ratios for comparable accuracy

Newmark time step/period $\Delta t_N/T$	Time-step ratio $\Delta t_W/\Delta t_N$
0·01	11·10
0·02	7·85
0·03	6·41
0·04	5·55
0·05	4·97
0·06	4·53
0·07	4·20
0·08	3·93
0·09	3·70
0·10	3·51

Comparing equations (56) and (71), it can be shown that for small Ω, comparable accuracy is obtained using the two methods if

$$\frac{N_N}{N_W} = \frac{\Delta t_W}{\Delta t_N} = \left(\frac{15}{\pi^2 (\Delta t_N/T)^2} \right)^{1/4} \tag{72}$$

where Δt_N and Δt_W are the time steps using Newmark's method and the proposed method respectively, and T is the undamped period. Thus, the ratio of time steps for comparable accuracy depends on the time step ratio $\Delta t_N/T$. Values of $\Delta t_W/\Delta t_N$ are summarized in Table III for various values of $\Delta t_N/T$ and have been verified by numerical experiments. A value of $\Delta t_N/T \leqslant 1/10$ has been suggested for MDOF systems,[22] where T is the period of the highest frequency which is considered important. Thus for similar accuracy, a time step 3·51 times greater than Newmark's method may be used with the proposed method if $\Delta t_N/T = 1/10$.

In determining the amplification matrix using the proposed method, it is necessary to solve the MDOF equivalent of equation (57), namely

$$\mathbf{P}^1 \mathbf{A} = -\mathbf{P}^0 \tag{73}$$

where $\mathbf{P}^0$ and $\mathbf{P}^1$ are $2N \times 2N$ symmetrical matrices and N is the number of degrees of freedom. This was achieved using $\mathbf{LDL}^T$ decomposition of $\mathbf{P}^1$, followed by the solution of $2N$ right-hand sides. To determine the amplification matrix of the single-step two-stage version of Newmark's method, it is necessary to carry out the following operations on $N \times N$ matrices:

(1) Invert two symmetrical matrices.
(2) Perform 10 matrix multiplications.
(3) Add scalar multiples of 13 matrices.

The two procedures were used to solve the free vibration of a cantilever, but banding of matrices was ignored, and hence the numerical problem was identical to that arising in a structure with full stiffness and mass matrices. Computer programs were written in FORTRAN 77, and CPU times were determined for various degrees of freedom, N. The programs were run on a SUN Enterprise 3000 computer. CPU times to determine t_N^A and t_W^A together with t^T, are summarized in

Table IV. Computer times in comparative studies

Total DOF	Time to generate amplification matrix (s)		Time per time-step (s)
N	$t_{\mathrm{N}}^{\mathrm{A}}$	$t_{\mathrm{W}}^{\mathrm{A}}$	t^{T}
100	13	7	0.021
200	109	68	0.109
300	402	237	0.264
400	922	658	0.634
500	2037	1225	0.962

Table IV. It may be seen in Table IV that the amplification matrix was determined more rapidly using the proposed method than using Newmark's method. In addition, as less time steps are required for similar accuracy, the time-stepping procedure requires less computational effort than Newmark's method. The numerical results for length of time step for similar accuracy were consistent with the theoretical values discussed above.

CONCLUSIONS

The time-stepping algorithm proposed in this paper is based on satisfying two general weighted integral equations. Coefficients arising in the procedure are selected to minimize an error term which simultaneously attempts to minimise period elongation and to preserve a spectral radius close to unity for small time steps. The unconditionally stable algorithm exhibits algorithmic damping of higher modes, with the amount of damping being readily adjusted as required. The algorithm exhibits a high degree of accuracy in terms of two commonly used measures, and for similar accuracy requires less computational effort than a widely used algorithm.

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